

# SEMI-INVARIANT SUBMANIFOLDS IN METRIC GEOMETRY OF AFFINORS

NOVAC-CLAUDIU CHIRIAC <sup>†</sup> AND MIRCEA CRASMAREANU

**ABSTRACT.** We introduce a generalization of structured manifolds as the most general Riemannian metric  $g$  associated to an affinor (tensor field of  $(1,1)$ -type)  $F$  and initiate a study of their semi-invariant submanifolds. These submanifolds are generalizations of CR-submanifolds of almost complex geometry and semi-invariant submanifolds of several interesting geometries (almost product, almost contact and others). We characterize the integrability of both invariant and anti-invariant distribution; the special case when  $F$  is covariant constant with respect to  $g$  gives major simplifications in computations.

2010 Mathematics Subject Classification: 53C40, 53C15, 53C12, 53C25.

Keywords:  $(g, F, \mu)$ -manifold: semi-invariant submanifold; (integrable) distribution.

## INTRODUCTION

The geometry of manifolds endowed with geometrical structures has been intensively studied and several important results have been published, see Yano-Kon [14]. The more important classes of such manifolds are formed by almost complex, almost product, almost contact, almost paracontact manifolds for which the cited book offers a good introduction. The geometry of submanifolds in these manifolds is very rich and interesting, as well, see for example the classical [7] or the more recent survey [8]. CR-submanifolds introduced by Bejancu in [2] (for almost complex geometry) respectively [5] (for almost contact geometry) had a great impact on the developing of the theory of submanifolds in these ambient manifolds; a proof of this fact is given by the books [4] and [13].

In the present paper we first introduce the concept of  $(g, F, \mu)$ -manifold which contains as particular cases all the above types of structures. Then we study semi-invariant submanifolds of a  $(g, F, \mu)$ -manifold, which are extensions of CR-submanifolds to this general class of manifolds. We find necessary and sufficient conditions for the integrability of both distributions on a semi-invariant submanifold, see Theorems 3.1 and 3.3. Particularly, we prove that some semi-invariant submanifolds carry a natural foliation, in Theorem 4.4 and we obtain characterizations of totally geodesic foliations on semi-invariant submanifolds in Theorems 4.8 and 4.10. For a particular value of the real parameter  $\mu$  we can connect our study with the almost symplectic geometry and this fact opens some possible further applications in physical sciences having as an example the relationship between CR-structures and Relativity pointed out in the last Chapter of [4].

This work is dedicated to Professor Aurel Bejancu on the occasion of his 65th birthday. His ideas represent a starting point for several important studies as the present Bibliography partially shows.

---

*Date:* August 19, 2011.

<sup>†</sup> Corresponding Author.

## 1. METRIC GEOMETRY OF AFFINORS AND SUBMANIFOLDS

Let  $M$  be an  $m$ -dimensional manifold for which we denote by  $C^\infty(M)$  the algebra of smooth functions on  $M$  and by  $\Gamma(TM)$  the  $C^\infty(M)$ -module of smooth sections of the tangent bundle  $TM$  of  $M$ ; let  $X, Y, Z, \dots$  denote such vector fields. We use the same notation  $\Gamma(V)$  for any other vector bundle  $V$  over  $M$ . Let also  $\mathcal{T}_1^1 M$  be the  $C^\infty(M)$ -module of  $\Gamma(TM \otimes T^*M)$  i.e. the real space of tensor fields of  $(1, 1)$ -type on  $M$ . Let consider a fixed  $F \in \mathcal{T}_1^1 M$  usually called *affinor* ([9]) or *vector 1-form*; the remarkable affinor of every manifold is the Kronecker tensor field  $I = (\delta_j^i)$ .

Fix  $\mu \in \{-1, +1\}$ . Let now  $g$  be a Riemannian metric on  $M$ .

**Definition 1.1**  $M$  is a  $(g, F, \mu)$ -manifold if :

$$g(FX, Y) + \mu g(X, FY) = 0. \quad (1.1)$$

The geometry of the data  $(M, g, F, \mu)$  is called *affinor-metric geometry*. If in particular,  $F_x$  is nondegenerate at any point  $x \in M$  then we say that  $M$  is a *nondegenerate  $(g, F, \mu)$ -manifold*; otherwise,  $M$  is called *degenerate  $(g, F, \mu)$ -manifold*.

The relation (1.1) says that the  $g$ -adjoint of  $F$  is  $F^* = -\mu F$ . In literature there is an abundance of examples of  $(g, F, \mu)$ -manifolds. Some of the main examples are presented here:

**Examples 1.2**

1. An *almost Hermitian manifold* ([4, p. 11])  $(M, g, J)$  is a nondegenerate  $(g, F, \mu = +1)$ -manifold; the nondegeneration is a consequence of  $J^2 = -I$ .
2. An *almost parahermitian manifold* ([1])  $(M, g, P)$  is a nondegenerate  $(g, F, \mu = +1)$ -manifold while an *almost Riemannian product manifold* is a nondegenerate  $(g, F, \mu = -1)$ -manifold; the nondegeneration is a consequence of  $P^2 = I$ .
3. An *almost contact metric manifold* ([4, p. 15])  $(M, g, \varphi, \xi, \eta)$  is a  $(g, F, \mu = +1)$ -manifold; as  $\varphi(\xi) = 0$ ,  $M$  is degenerate.
4. An *almost paracontact manifold* ([12])  $(M, g, \varphi, \xi, \eta)$  is a  $(g, F, \mu = +1)$ -manifold. As in the previous example we have  $\varphi(\xi) = 0$  and therefore  $M$  is degenerate.
5. The general case of a nondegenerate  $(g, F, \mu = +1)$ -manifold is called *structured manifold* in [11].

Recall that a real  $2m$ -dimensional manifold  $M$  is called an *almost symplectic manifold* if it is endowed with a nondegenerate 2-form  $\Omega \in \Lambda^2(M)$ . We derive the following characterization:

**Proposition 1.3** *Let  $M$  be a  $(g, F, \mu = +1)$ -manifold. Then  $M$  is nondegenerate if and only if  $\Omega$  defined by:*

$$\Omega(X, Y) = g(FX, Y) \quad (1.2)$$

*is an almost symplectic structure. In this case  $m$  is even.*

**Proof**  $\Omega$  is skew-symmetric from  $\mu = +1$ . A straightforward computation yields that  $\Omega$  is nondegenerate if and only if  $M$  is a nondegenerate  $(g, F, \mu = +1)$ -manifold.  $\square$

**Example 1.4** For Example 1.2.1  $\Omega$  is exactly the *fundamental* or *Kähler* 2-form and then inspired by this fact we introduce:

**Definition 1.5** For a nondegenerate  $(g, F, \mu = +1)$ -manifold  $\Omega$  is call *the fundamental 2-form*.

In the last part of this section let us recall briefly the geometry of Riemannian submanifolds. Consider an  $n$ -dimensional submanifold  $N$  of  $M$ . Then the main objects induced by the Levi-Civita connection  $\tilde{\nabla}$  of  $(M, g)$  on  $N$  are involved in the well known Gauss-Weingarten equations:

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad (1.3)$$

for any  $X, Y \in \Gamma(TN)$  and  $V \in \Gamma(T^\perp N)$ . Here  $\nabla$  is the Levi-Civita connection on  $N$ ,  $h$  is the second fundamental form of  $N$ ,  $A_V$  is the Weingarten operator with respect to the normal section  $V$  and  $\nabla^\perp$  is the normal connection in the normal bundle  $T^\perp N$  of  $N$ . Let us point out that  $h$  and  $A_V$  are related by:

$$g(h(X, Y), V) = g(A_V X, Y). \quad (1.4)$$

If  $h$  vanishes identically on  $N$  then  $N$  is called *totally geodesic*.

## 2. SUBMANIFOLDS IN AFFINOR-METRIC GEOMETRY

Next, we consider a submanifold  $N$  of a  $(g, F, \mu)$ -manifold  $M$ . Then  $g$  induces a Riemannian metric on  $N$  which we denote by the same symbol  $g$ . Then, following the definition given by Bejancu [2] for CR-submanifolds we introduce a special class of submanifolds of  $M$  as follows:

**Definition 2.1**  $N$  is a *semi-invariant submanifold* of  $M$  if there exists a distribution  $D$  on  $N$  satisfying the conditions:

(i)  $D$  is  $F$ -invariant:

$$F(D_x) \subset D_x, \quad \forall x \in N. \quad (2.1)$$

(ii) The complementary orthogonal distribution  $D^\perp$  to  $D$  in  $TN$  is  $F$ -anti-invariant, that is:

$$F(D_x^\perp) \subset T_x^\perp N, \quad \forall x \in N. \quad (2.2)$$

(iii)  $F^2(D^\perp)$  is a distribution on  $N$ .

Some particular classes of semi-invariant submanifolds are defined as follows. Let  $p$  and  $q$  be the ranks of the distributions  $D$  and  $D^\perp$  respectively. If  $q = 0$ , that is  $D^\perp = \{0\}$ , we say that  $N$  is an  *$F$ -invariant submanifold* of  $M$ . If  $p = 0$ , that is  $D = \{0\}$ , we call  $N$  an  *$F$ -anti-invariant submanifold* of  $M$ .

If  $pq \neq 0$  then  $N$  is called a *proper* semi-invariant submanifold. Now, we denote by  $\tilde{D}$  the complementary orthogonal vector bundle to  $F(D^\perp)$  in  $T^\perp N$ . If  $\tilde{D} = \{0\}$  then we say that  $N$  is a *normal  $F$ -semi-invariant submanifold*.

Thus,  $N$  is an  $F$ -invariant, respectively  $F$ -anti-invariant, if and only if:

$$F(TN) \subset TN \quad (\text{resp. } F(TN) \subset T^\perp N). \quad (2.3)$$

$N$  is normal  $F$ -semi-invariant if and only if:

$$F(D^\perp) = T^\perp N. \quad (2.4)$$

### Examples 2.2

- 1) For Example 1.2.1 we obtain the classical concept of CR-submanifold of Bejancu [4, p. 20]; the condition iii) is satisfied from  $J^2 = -I$ .
- 2) For Example 1.2.2 we obtain the notion of semi-invariant submanifold; for the almost parahermitian case see [1] while for the second case see [3]. The condition iii) is satisfied again from  $P^2 = -I$ .
- 3) For Example 1.2.3 we obtain the notion of semi-submanifold [4, p. 100] with  $\xi \in T^\perp N$ . This last condition implies  $TN \subset \ker \eta$  and since  $\varphi|_{\ker \eta}$  is an almost complex structure we

get iii).

4) For Example 1.2.4 we obtain the concept of semi-submanifold from [10] with  $\xi \in T^\perp N$ . Again this condition means  $TN \subset \ker \eta$  and since  $\varphi|_{\ker \eta}$  is an almost product structure we have iii).

5) The condition (iii) does not appears in [11].

Returning to the Definition 2.1 we deduce that the tangent bundle  $TN$  and the normal bundle  $T^\perp N$  of a semi-invariant submanifold  $N$  have the orthogonal decompositions:

$$TN = D \oplus D^\perp, \quad T^\perp N = F(D^\perp) \oplus \tilde{D}. \quad (2.5)$$

Then we denote by  $P$  and  $Q$  the projection morphisms of  $TN$  on  $D$  and  $D^\perp$  respectively and obtain for  $X = PX + QX \in \Gamma(TN)$ :

$$FX = \varphi X + \omega X \quad (2.6)$$

where we put:

$$\varphi = F \circ P, \quad \omega = F \circ Q. \quad (2.7)$$

Thus  $\varphi$  is a tensor field of  $(1,1)$ -type on  $N$  while  $\omega$  is a  $F(D^\perp)$ -valued vector 1-form on  $N$ . Thus we derive:

**Proposition 2.5** *Let  $N$  be a semi-invariant submanifold of a  $(g, F, \mu)$ -manifold  $M$ . Then:*

(iv)  *$N$  is a  $(g, \varphi, \mu)$ -manifold.*

(v)  *$F^2(D^\perp)$  is a vector subbundle of  $D^\perp$ .*

(vi) *The vector bundle  $\tilde{D}$  is  $F$ -invariant i.e. for all  $x \in N$  we have:  $F(\tilde{D}_x) \subset \tilde{D}_x$ .*

**Proof** (iv) By definition,  $g$  is a Riemannian metric on  $N$  and  $\varphi$  is a tensor field of  $(1,1)$ -type on  $N$ ; we need only to show (2.1). By using (1.1) for  $F$  we obtain for  $X, Y \in \Gamma(TN)$ :

$$\begin{aligned} g(\varphi X, Y) &= g(FPX, Y) = g(FPX, PY) = -\mu g(PX, FPY) = \\ &= -\mu g(X, FPY) = -\mu g(X, \varphi Y). \end{aligned}$$

(v) Take  $X \in \Gamma(D)$  and  $Y \in \Gamma(D^\perp)$  in (2.1):  $g(X, F^2Y) = -\mu g(FX, FY) = 0$  since  $FX \in \Gamma(D)$  and  $FY \in \Gamma(T^\perp N)$ . Hence  $F^2(D^\perp)$  is orthogonal to  $D$  and by condition (iii) we deduce that  $F^2(D^\perp)$  is a vector subbundle of  $D^\perp$ .

(vi) Take  $X \in \Gamma(TN)$ ,  $Y \in \Gamma(D^\perp)$  and  $V \in \Gamma(\tilde{D})$ . Then we obtain:

$$g(FV, X) = -\mu g(V, FX) = -\mu g(V, \varphi X + \omega X) = 0$$

and:

$$g(FV, FY) = -\mu g(V, F^2Y) = 0$$

since  $\varphi X \in \Gamma(D)$ ,  $\omega X \in \Gamma(FD^\perp)$  and  $F^2Y \in \Gamma(D^\perp)$ . Thus  $F\tilde{D}$  is orthogonal to  $TN \oplus FD^\perp$ , that is  $F\tilde{D}$  is a vector subbundle of  $\tilde{D}$ . This completes the proof of the proposition.  $\square$

In the non-degenerated case we have equalities for the above inclusions:

**Corollary 2.6** *Let  $N$  be a semi-invariant submanifold of a nondegenerate  $(g, F, \mu)$ -manifold  $M$ . Then:*

1) *the above distributions satisfy:*

$$F(D) = D, \quad F^2(D^\perp) = D^\perp, \quad F(\tilde{D}) = \tilde{D}. \quad (2.8)$$

2) *if  $\mu = +1$  then  $D^\perp$  and  $F(D^\perp)$  are Lagrangian distribution on  $(TM, \Omega)$ . In particular if  $N$  is a normal semi-invariant submanifold then  $T^\perp N$  is a Lagrangian submanifold in  $(TM, \Omega)$ .*

**Proof** We need to prove only 2).

2.1) Let  $X, Y \in \Gamma(D^\perp)$ ; then  $\Omega(X, Y) = g(FX, Y) = 0$  since  $FX \in \Gamma(T^\perp N)$  while  $Y \in \Gamma(TN)$ .

2.2) Let  $X, Y \in \Gamma(F(D^\perp))$ ; then  $\Omega(X, Y) = g(FX, Y) = 0$  since  $FX \in \Gamma(TN)$  while  $Y \in \Gamma(T^\perp N)$ .  $\square$

The second part of the above Corollary is extremely important since it relates the geometry of semi-invariant submanifolds with the almost symplectic geometry, a topic very studied from the point of view of applications in Analytical Mechanics.

### 3. INTEGRABILITY OF DISTRIBUTIONS ON A SEMI-INVARIANT SUBMANIFOLD

Let  $N$  be a semi-invariant submanifold of a  $(g, F, \mu)$ -manifold  $M$ . Then we recall that the Nijenhuis tensor field of  $F$  is defined as follows ([4, p.11]):

$$N_F(X, Y) = [FX, FY] + F^2[X, Y] - F[FX, Y] - F[X, FY], \quad (3.1)$$

for any  $X, Y \in \Gamma(TM)$ . In a similar way, the Nijenhuis tensor field of  $\varphi$  on  $N$  is given by:

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y], \quad (3.2)$$

for any  $X, Y \in \Gamma(TN)$ . We recall that a tensor field of  $(1, 1)$ -type defines an *integrable structure* on a manifold if and only if its Nijenhuis tensor field vanishes identically on the manifold. Now we obtain necessary and sufficient conditions for the integrability of  $D$  and  $D^\perp$  in terms of Nijenhuis tensor fields of  $F$  and  $\varphi$ .

**Theorem 3.1** *Let  $N$  be a semi-invariant submanifold of a  $(g, F, \mu)$ -manifold  $M$ . Then the following assertions are equivalent:*

- 1)  $D$  is an integrable distribution.
- 2) The Nijenhuis tensor field of  $\varphi$  satisfies:

$$Q \circ N_\varphi = 0, \quad \forall X, Y \in \Gamma(D). \quad (3.3)$$

- 3) The Nijenhuis tensor fields of  $F$  and  $\varphi$  satisfy the equality:  $N_F = N_\varphi$  on  $D$ .

**Proof** Firstly, we note that  $D$  is integrable if and only if:

$$Q([X, Y]) = 0, \quad \forall X, Y \in \Gamma(D). \quad (3.4)$$

Since the last three terms in the right side of (3.2) lie in  $\Gamma(D)$  we deduce that:

$$Q \circ N_\varphi(X, Y) = Q([FX, FY]), \quad \forall X, Y \in \Gamma(D). \quad (3.5)$$

As  $M$  is nondegenerate we deduce that  $\varphi$  is an automorphism on  $\Gamma(D)$ . Thus the equivalence of 1) and 2) follows directly. Next, we obtain for any  $X, Y \in \Gamma(D)$ :

$$N_F(X, Y) = N_\varphi(X, Y) + F\omega([X, Y]) - \omega([\varphi X, Y]) - \omega([X, \varphi Y]). \quad (3.6)$$

If  $D$  is integrable then the last three terms of (3.6) vanishes and this yields 3). Conversely, suppose that  $N_F = N_\varphi$  on  $D$ ; then:

$$F\omega([X, Y]) = \omega([\varphi X, Y] + [X, \varphi Y]). \quad (3.7)$$

Obviously the right-hand-side of the previous equation is in  $\Gamma(F(D^\perp)) \subset \Gamma(T^{bot}N)$ . On the other hand, the left-hand-side is in  $\Gamma(F^2D^\perp) \subset \Gamma(TN)$ ; we conclude that both sides in (3.7) must vanish.

Finally, from:  $F^2Q([X, Y]) = 0$  and  $F^2$  automorphism of  $\Gamma(TM)$  we deduce 1).  $\square$

**Remark 3.2** For Example 1.2.1 the equivalence of 1) and 2) is exactly the Theorem 2.2. of [4, p. 25] while the equivalence of 1) and 3) is the Theorem 2.1. of [4, p. 25].

Now, we consider  $X, Y \in \Gamma(D^\perp)$ . Then taking into account that  $\varphi X = \varphi Y = 0$  we get:

$$N_\varphi(X, Y) = F^2 P[X, Y] \quad (3.8)$$

and this enables us to state the following:

**Theorem 3.3** *Let  $N$  be a semi-invariant submanifold of a nondegenerate  $(g, F, \mu)$ -manifold. Then  $D^\perp$  is integrable if and only if the Nijenhuis tensor field of  $\varphi$  vanishes identically on  $D^\perp$ .*

**Remark 3.4** For Example 1.2.1 the above result is the Theorem 2.3. of [4, p. 26].

#### 4. A NATURAL FOLIATION ON A SEMI-INVARIANT SUBMANIFOLD

Let  $\tilde{\nabla}$  be the Levi-Civita connection on  $M$  with respect to the Riemannian metric  $g$ . Then  $F$  is a *parallel tensor field* on  $M$  if:

$$\tilde{\nabla} F = 0. \quad (4.1)$$

##### Examples 4.1

- 1) For Example 1.2.1 we have the notion of *Kähler manifold*.
- 2) For Example 1.2.2, in the first part we have the concept of *para-Kähler manifold* while for the second part the notion of *locally Riemannian product manifold*.
- 3) For Example 1.2.3 we get the notion of *cosymplectic manifold*.

In the present section we study the geometry of semi-invariant submanifolds of  $(g, F, \mu)$ -manifolds with parallel tensor field  $F$ . First, we prove the following:

**Proposition 4.2** *Let  $N$  be a semi-invariant submanifold of a nondegenerate  $(g, F, \mu)$ -manifold with parallel tensor field  $F$ . Then for all  $X, Y \in \Gamma(D^\perp)$ :*

$$A_{FX}Y - A_{FY}X = \varphi([X, Y]). \quad (4.2)$$

**Proof** By using the Weingarten equation and the parallelism condition we get:

$$A_{FX}Y = \nabla_Y^\perp FX - \nabla_Y FX = \nabla_Y^\perp FX - F(\tilde{\nabla}_X Y). \quad (4.2)$$

Writing a similar equation by interchanging  $X$  and  $Y$  and then subtracting we obtain:

$$A_{FX}Y - A_{FY}X = \nabla_Y^\perp FX - \nabla_X^\perp FY + F([X, Y]), \quad (4.3)$$

since  $\nabla$  is a torsion-free linear connection. Thus (4.2) is obtained by equalizing the tangent parts to  $N$  in the above equation.  $\square$

**Example 4.3** The relation (4.2) becomes for Example 1.2.1 the equation (2.2) of [4, p. 43].

Now, we can state the following main result:

**Theorem 4.4** *Let  $N$  be a semi-invariant submanifold of a nondegenerate  $(g, F, \mu = +1)$ -manifold with parallel tensor field  $F$ . Then the  $F$ -anti-invariant distribution  $D^\perp$  is integrable.*

**Proof** For any  $X, Y \in \Gamma(D^\perp)$  and  $Z \in \Gamma(D)$  we have:

$$\begin{aligned} g(A_{FX}Y, Z) &= -g(F\tilde{\nabla}_Y X, Z) = +\mu g(\tilde{\nabla}_Y X, FZ) = -\mu g(X, \tilde{\nabla}_Y FZ) = \\ &= \mu^2 g(FX, \tilde{\nabla}_Y Z) = \mu^2 g(FX, [Y, Z] + \tilde{\nabla}_Z Y) = \mu^2 g(FX, \tilde{\nabla}_Z Y). \end{aligned} \quad (4.4)$$

Also, we have:

$$g(A_{FY}X, Z) = \mu^2 g(FY, \tilde{\nabla}_Z X) = -\mu^2 g(F\tilde{\nabla}_Z Y, X) = \mu^3 g(\tilde{\nabla}_Z Y, FX). \quad (4.5)$$

Comparing (4.4) and (4.5) we deduce that for  $\mu = +1$ :

$$g(A_{FX}Y - A_{FY}X, Z) = 0$$

which means that  $A_{FX}Y - A_{FY}X \in \Gamma(D^\perp)$ . On the other hand, from (4.2) we conclude that:

$$A_{FX}Y - A_{FY}X \in \Gamma(D).$$

and thus we have that:

$$A_{FX}Y - A_{FY}X = 0. \quad (4.6)$$

Finally, returning to (4.2) and taking into account that  $F$  is nondegenerate we deduce that:

$$P[X, Y] = 0,$$

that is,  $D^\perp$  is integrable.  $\square$

**Remark 4.5** For Example 1.2.1 the above result is part (i) of Theorem 1.1. of [4, p. 39].

Regarding the integrability of  $D$  we prove the following:

**Theorem 4.6** *Let  $N$  be a semi-invariant submanifold of a nondegenerate  $(g, F, \mu)$ -manifold  $M$  with parallel tensor field  $F$ . Then the  $F$ -invariant distribution  $D$  is integrable if and only if the second fundamental form  $h$  of  $N$  satisfies for any  $X, Y \in \Gamma(D)$  and  $Z \in \Gamma(D^\perp)$ :*

$$g(h(X, \varphi Y) - h(Y, \varphi X), FZ) = 0. \quad (4.7)$$

**Proof** By using the Gauss equation we deduce that:

$$\nabla_X \varphi Y + h(X, \varphi Y) = \varphi(\nabla_X Y) + \omega(\nabla_X Y) + Fh(X, Y). \quad (4.8)$$

Write a similar equation by interchanging  $X$  and  $Y$ , and then subtracting we obtain:

$$\nabla_X \varphi Y - \nabla_Y \varphi X + h(X, \varphi Y) - h(Y, \varphi X) = \varphi([X, Y]) + \omega([X, Y]) \quad (4.9)$$

since  $h$  is symmetric and  $\nabla$  is a torsion-free linear connection. Equalize the normal parts in the above equation and obtain:

$$h(X, \varphi Y) - h(Y, \varphi X) = \omega([X, Y]). \quad (4.10)$$

Now, suppose that  $D$  is integrable; then (4.7) is immediately. Conversely, if (4.6) is satisfied, then with (4.10) we deduce that:

$$0 = -\mu g(Q[X, Y], F^2 Z).$$

Since  $F$  is nondegenerate we infer that  $F^2$  is an automorphism of  $\Gamma(D^\perp)$  and hence  $D$  is integrable.  $\square$

**Remark 4.7** In particular, if  $F$  is an almost complex structure on  $M$  then we obtain the results of Bejancu [4] and Blair-Chen [6] respectively, for CR-submanifolds.

Now, for  $\mu = +1$  we denote by  $\mathcal{F}^\perp$  the natural foliation defined by the  $F$ -anti-invariant distribution  $D^\perp$  and call it the  $F$ -anti-invariant foliation on  $N$ . We recall that  $\mathcal{F}^\perp$  is called a *totally geodesic foliation* if each leaf of  $\mathcal{F}^\perp$  is totally geodesic immersed in  $N$ . Thus  $\mathcal{F}^\perp$  is totally geodesic if and only if the Levi-Civita connection  $\nabla$  of  $N$  satisfies for all  $Y, Z \in \Gamma(D^\perp)$ :

$$\nabla_Y Z \in \Gamma(D^\perp). \quad (4.11)$$

**Theorem 4.8** *Let  $N$  be a semi-invariant submanifold of a nondegenerate  $(g, F, \mu)$ -manifold  $M$  with parallel tensor field  $F$ . Then the following assertions are equivalent:*

- (i) *The  $F$ -anti-invariant foliation is totally geodesic.*
- (ii) *The second fundamental form  $h$  of  $N$  satisfies for all  $X \in \Gamma(D)$  and  $Y \in \Gamma(D^\perp)$ :*

$$h(X, Y) \in \Gamma(\tilde{D}). \quad (4.12)$$

- (iii)  *$D^\perp$  is  $A_V$ -invariant for any  $V \in \Gamma(FD^\perp)$  that is we have for all  $Y \in \Gamma(D^\perp)$ :*

$$A_V Y \in \Gamma(D^\perp). \quad (4.13)$$

**Proof** We have for any  $X \in \Gamma(D)$  and  $Y, Z \in \Gamma(D^\perp)$ :

$$\begin{aligned} g(\nabla_Y Z, FX) &= g(\tilde{\nabla}_Y Z, FX) = -\mu g(\tilde{\nabla}_Y FZ, X) = \\ &= \mu g(A_{FZ} Y, X) = \mu g(h(X, Y), FZ). \end{aligned} \quad (4.14)$$

Now, suppose that  $\mathcal{F}^\perp$  is totally geodesic; then the first term of (4.14) vanishes. Hence the last term in (4.14) vanishes which implies ii). Conversely, suppose (4.12) is satisfied. Then from (4.14) we deduce (4.11) since  $F$  is an automorphism of  $\Gamma(D)$ . This proves the equivalence of (i) and (ii). The equivalence of (ii) and (iii) is straightforward.  $\square$

**Remark 4.9** For Example 1.2.1 the equivalence of (i) and (ii) is the Theorem 1.3. of [4, p. 41].

Finally, we can prove the following:

**Theorem 4.10** *Let  $N$  be a semi-invariant submanifold of a nondegenerate  $(g, F, \mu)$ -manifold with parallel tensor field  $F$ . Then the  $F$ -invariant distribution  $D$  is integrable and the foliation  $\mathcal{F}$  defined by  $D$  is totally geodesic if and only if the second fundamental form  $h$  of  $N$  satisfies for all  $X, Y \in \Gamma(D)$ :*

$$h(X, Y) \in \Gamma(\tilde{D}). \quad (4.15)$$

**Proof**  $D$  is integrable and  $\mathcal{F}$  is totally geodesic if and only if for all  $X, U \in \Gamma(D)$ :

$$\nabla_X U \in \Gamma(D). \quad (4.16)$$

This is equivalent to:

$$g(\tilde{\nabla}_X U, Z) = 0, \quad (4.17)$$

for all  $Z \in \Gamma(D^\perp)$ . As  $F$  is an automorphism of  $\Gamma(D)$  we can write the above equality as follows:

$$g(\tilde{\nabla}_X FY, Z) = 0, \quad (4.18)$$

for all  $X, Y \in \Gamma(D)$  and  $Z \in \Gamma(D^\perp)$ , which is equivalent to:

$$g(\tilde{\nabla}_X Y, FZ) = 0. \quad (4.19)$$

By using the Gauss equation, the last relation is equivalent to:

$$g(h(X, Y), FZ) = 0, \quad (4.20)$$

which completes the proof of the theorem.  $\square$

**Remark 4.11** For Example 1.2.1 the above result is Theorem 1.2. of [4, p. 40].



## REFERENCES

- [1] C. L. Bejan, *CR-Submanifolds of hyperbolic almost Hermitian manifolds*, Demonstratio Mathematica, 23(1990), 335-343. MR1101496 (92b:53036).
- [2] A. Bejancu, *CR-Submanifolds of a Kaehler manifold I*, Proc. Amer. Math. Soc., 69(1978), 134-142. MR0467630 (57 #7486).
- [3] A. Bejancu, *Semi-invariant submanifolds of locally product Riemannian manifolds*, An. Univ. Timisoara Ser. Stiint. Mat., 22(1984), no. 1-2, 3-11. MR0767127 (86h:53037)
- [4] A. Bejancu, *Geometry of CR-Submanifolds*, D. Reidel Publish. Comp., Dordrecht, 1986. MR0861408 (87k:53126).
- [5] A. Bejancu and N. Papaghiuc, *Semi-invariant submanifolds of a Sasakian manifold*, An. St. Univ. "Al. I. Cuza" Iasi, 27(1981), 163-170. MR0618723 (82i:53037).
- [6] D. E. Blair and B. Y. Chen, *On CR-submanifolds of Hermitian manifolds*, Israel J. Math., 34(1979), 353-363. MR0570892 (81f:53049).
- [7] B. Y. Chen, *Geometry of Submanifolds and Its Applications*, Science Univ. Tokyo, 1981. MR0627323 (82m:53051).
- [8] B.Y. Chen, *Riemannian submanifolds*. Handbook of differential geometry, Vol. I, 187-418, North-Holland, Amsterdam, 2000. MR1736854 (2001b:53064)
- [9] M. Doupovec and I. Kolár, *Natural affinors on time-dependent Weil bundles*, Arch. Math. (Brno), 27B (1991), 205-209. MR1189217 (93j:58004)
- [10] S. Ianus and I. Mihai, *Semi-invariant submanifolds of an almost paracontact manifolds*, Tensor, 39(1982), 195-200. MR0836935 (87c:53102).
- [11] L. Ornea, *CR-submanifolds. A class of examples*, Rev. Roumaine Math. Pures Appl., 51(2006), no. 1, 77-85. MR2275320 (2007i:53057) ([http://arxiv.org/PS\\_cache/math/pdf/0505/0505150v1.pdf](http://arxiv.org/PS_cache/math/pdf/0505/0505150v1.pdf))
- [12] I. Sato, *On a structure similar to the almost contact structure*, Tensor, 30(1976), 219-224. MR0442845 (56 #1221).
- [13] K. Yano and M. Kon, *CR-Submanifolds of Kaehlerian and Sasakian Manifolds*, Birkhäuser, Boston, 1983. MR0688816 (84k:53051).
- [14] K. Yano and M. Kon, *Structures on Manifolds*, World Scientific, Singapore, 1984. MR0794310 (86g:53001).

(N.-C. Chiriac) DEPARTMENT OF MATHEMATICS, UNIVERSITY CONSTANTIN BRÂNCUSI, BD. REPUBLICII, NR. 1, TÂRGU-JIU, 210152, ROMANIA

*E-mail address:* novac (at) utgjiu.ro

(M. Crasmareanu) FACULTY OF MATHEMATICS, 'AL. I. CUZA' UNIVERSITY OF IASI, BD. CAROL I, NR. 11, IASI, 700506, ROMANIA, [HTTP://WWW.MATH.UAIC.RO/~MCRASM](http://www.math.uaic.ro/~mcrasm)

*E-mail address:* mcrasm (at) uaic.ro